Notes on Menger Curvature

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Update (Sept. 2003):

The number $K(x_1, x_2, x_3) = \frac{1}{R}$, where R is the radius of the *circumcircle* through the three points x_1, x_2, x_3, is called the Menger curvature of the triple [3]. The derivation of the formula below, then relies on using the well known relation between the area of the triangle, A, through these three points and R: $K = \frac{1}{R} = \frac{4A}{abc}$, where a, b and c are the length of the sides of the triangle. Note that Heron's formula gives us: $16A^2 = (a+b+c)(b+c-a)(a+c-a)(a+b-c)$, and this directly give the formula below. An alternative formula is provided by the Cayley-Menger determinant [1], which avoids directly computing the side lengths (as square roots): $16A^2 = \left|a^2(a^2-b^2-c^2) + b^2(b^2-a^2-c^2) + c^2(c^2-a^2-b^2)\right|$.

Let f be a regular curve of class C^2 in a Euclidean space, E^n. Let x_1, x_2, x_3 be distinct points of f. Let the length of "side" vectors through each pair of points be denote by $x_i x_j = |x_j - x_i|$. Then, define [1, vol.1, p.273]:

$$K(x_1, x_2, x_3) = \frac{\sqrt{(x_1x_2 + x_2x_3 + x_3x_1)(x_1x_2 + x_2x_3 - x_3x_1)(x_1x_2 - x_2x_3 + x_3x_1)(x_1x_2 - x_2x_3 - x_3x_1)}}{x_1x_2 \cdot x_2x_3 \cdot x_3x_1},$$

where the (non-negative) number K is called *Menger's* curvature. As x_2 and x_3 approach x_1 on f, $K(x_1, x_2, x_3)$ tends towards the *curvature* of f at x_1. Also, $K = 0$ if and only if x_1, x_2 and x_3 are *collinear*.

In the complex domain, for $z_1, z_2, z_3 \in C$, this notion is called the Menger-Melnikov curvature [2]:

$$c(z_1, z_2, z_3)^2 = \sum_\sigma \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})\overline{(z_{\sigma(2)} - z_{\sigma(3)})}} \quad ,$$

where the sum is taken over all permutations of σ of $\{1, 2, 3\}$. This identity is transformed for 1-sets in E^n to:

$$c(x_1, x_2, x_3)^2 = \sum_\sigma \frac{(x_{\sigma(1)} - x_{\sigma(3)}) \cdot (x_{\sigma(2)} - x_{\sigma(3)})}{\left|x_{\sigma(1)} - x_{\sigma(3)}\right|^2 \left|x_{\sigma(2)} - x_{\sigma(3)}\right|^2} \quad ,$$

or equivalently, after some manipulations:

$$c(x_1, x_2, x_3)^2 = 4 \left\{ \frac{|x_1 - x_3|^2 \, |x_2 - x_3|^2 - ((x_1 - x_3) \cdot (x_2 - x_3))^2}{|x_1 - x_3|^2 \, |x_2 - x_3|^2 \, |x_1 - x_2|^2} \right\} \quad ,$$

which, by Schwartz inequality, can be shown to always be non-negative.

References

[1] Marcel Berger. *Geometry*. Universitext. Springer-Verlag, Berlin. Translation by M.Cole and S.Levy of the French 1977 edition. 2 volumes.

[2] Hany M. Farag. Curvatures of the Melnikov type, Hausdorff dimension, rectifiability, and singular integrals on r^n. *Pacific Journal of Math.*, 196(2):317–339, 2000.

[3] J. C. Léger. Menger curvature and rectifiability. *Annals of Mathematics*, 149:831–869, 1999.