# Notes on Quadratics, Cubics and Quartics

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July 2000

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# Chapter 1

# Solving Quadratics

Consider the classical quadratic equation:

$$a\,x^2 + b\,x + c = 0 \ ,$$

with real coefficients $a \neq 0, b, c$. Then, the usual formula giving the roots of this equation in terms these coefficients are:

$$\begin{aligned} x &= \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right) \\ &= \frac{1}{2c\left(-b \mp \sqrt{b^2 - 4ac}\right)}. \end{aligned}$$

Unfortunately, if either $a$ or $c$ (or both) are small, then the subtraction of $b$ from the "discriminant",[1] $b^2 - 4ac$, will give a small number, and its division by $2a$ will be numerically unstable [6, p.184].

The correct way to compute the root is to, evaluate:

$$q = -\frac{1}{2}\left[b + sgn(b)\sqrt{b^2 - 4ac}\right] ,$$

where, the two roots are obtained as:

$$x_1 = \frac{q}{a} \ \ and \ \ x_2 = \frac{c}{q}.$$

[1] Strictly speaking, for a quadratic, the discriminant is given as: $D = (b^2 - 4ac)/(2a)^2$.

# Chapter 2

# Solving Cubics

## 2.1 Algebraic Solution

Consider the cubic implicit polynomial in terms of $\rho_y$ :

$$p^{(3)}(\rho_y) = \phi_3\,{\rho_y}^3 + \phi_2\,{\rho_y}^2 + \phi_1\,\rho_y + \phi_0 = 0,$$

where $\phi_3 \neq 0$.[1] In order to solve the above cubic, we first reduce it to a simpler form [1, §4.4], by first dividing this cubic by $\phi_3$, and then performing the (Viète) substitution:

$$\Upsilon = \rho_y + \frac{\phi_2}{3\phi_3}\ ,$$

to get the simplified expression:[2]

$$\Upsilon^3 + \rho\,\Upsilon + \sigma = 0\ , \tag{2.1}$$

where:

$$\begin{aligned} \rho &= \frac{\phi_1}{\phi_3} - \frac{\phi_2^2}{3\phi_3^2} \\ \sigma &= \frac{\phi_0}{\phi_3} - \frac{\phi_1\phi_2}{3\phi_3^2} + \frac{2\phi_2^3}{27\phi_3^3} \end{aligned}$$

Two special cases can be easily solved. If $\rho = 0$, then the solution is immediate, *i.e.*: $\Upsilon = -\sigma^{1/3}$. If $\sigma = 0$, we have one root $\Upsilon = 0$, and two roots $\Upsilon = \pm\sqrt{-\rho}$. Otherwise, in the more general case, various substitutions are used to find the three (real) roots of this cubic. For example, we may perform the (yet another of Viète's) substitution [1, §5.5]:[3]

$$\Upsilon = \overline{\Upsilon} - \frac{\rho}{3\overline{\Upsilon}},$$

and multiply the result by $\overline{\Upsilon}^3$ in order to get a (tri)quadratic in $\overline{\Upsilon}^3$:

$$\left(\overline{\Upsilon}^3\right)^2 + \sigma\,\overline{\Upsilon}^3 - \frac{\rho^3}{27} = 0.$$

Then, using completion of the square leads to:

[1] Note that, if $\phi_3$ is close to zero, according to some tolerance level, this equation may be considered as a quadratic, *i.e.*, by fixing $\phi_3 = 0$.

[2] Where the roots of this equation differ from the original by the factor $\phi_2/(3\phi_3)$ .

[3] See [5, p.47] for a different (the original 16th century) derivation of the same result.

$$\overline{\Upsilon}^3 = -\frac{\sigma}{2} \pm \sqrt{\frac{\sigma^2}{4} + \frac{\rho^3}{27}}. \tag{2.2}$$

This gives *six* solutions for $\overline{\Upsilon}$ in the form of cube roots. Substituting back into $\Upsilon = \overline{\Upsilon} - (\rho/3\overline{\Upsilon})$, we get three pairs of solutions for $\Upsilon$ (and thus $\rho_y$), paired solutions being equal. This result is also known as Cardano's solution [5, p.47].[4]

Furthermore, in terms of the original coefficients, we have for the above cubic in $\overline{\Upsilon}$:

$$\frac{\sigma^2}{4} + \frac{\rho^3}{27} = \frac{1}{108\,\phi_3^4}\left(27\phi_0^2\phi_3^2 - 18\phi_0\phi_1\phi_2\phi_3 + 4\phi_0\phi_2^3 - \phi_1^2\phi_2^2 + 4\phi_1^3\phi_3\right),$$

and thus:

$$\overline{\Upsilon}^3 = \left(\frac{\phi_0}{\phi_3} - \frac{\phi_1\phi_2}{3\phi_3^2} + \frac{2\phi_2^3}{27\phi_3^3}\right) \pm \frac{1}{6\sqrt{3}\phi_3^2}\sqrt{27\phi_0^2\phi_3^2 - 18\phi_0\phi_1\phi_2\phi_3 + 4\phi_0\phi_2^3 - \phi_1^2\phi_2^2 + 4\phi_1^3\phi_3}.$$

It proves useful to evaluate the so-called "discriminant" of the above cubic in $\overline{\Upsilon}$.[5] This is define as:

$$D = \prod \left(\overline{\Upsilon_i} - \overline{\Upsilon_j}\right)^2 = \left(\overline{\Upsilon_1} - \overline{\Upsilon_2}\right)^2 \left(\overline{\Upsilon_2} - \overline{\Upsilon_3}\right)^2 \left(\overline{\Upsilon_1} - \overline{\Upsilon_3}\right)^2.$$

One can show then that [1, p.120]:[6]

$$D = -4\rho^3 - 27\sigma^2,$$

which can be used to simplify eqn. (2.2) to:

$$\overline{\Upsilon}^3 = -\frac{\sigma}{2} \pm \frac{1}{6}\sqrt{\frac{-D}{3}}.$$

Therefore we have a test for the number of type of roots of our cubic:

**Theorem 2.1.1 (Discriminant of a cubic)** *A cubic (or quadratic) equation with real coefficient has three real roots if its discriminant is nonnegative ($D \geq 0$), and two imaginary roots if its discriminant is negative ($D < 0$).*

Note that if $D = 0$ then we have multiple (real) roots, where at least two roots are equal. If $D > 0$ then we have real distinct roots, but our formula above expresses them via complex numbers, an oddity which cannot be alleviated [1, p.120].[7] In particular, we may end-up with non-null imaginary parts which do not cancel-up, when dealing with small coefficients, due to numerical instabilities. It therefore becomes interesting to find an alternative solution to the above "classical" method based on radicals.

### 2.1.1 Trigonometric solution to the cubic equation

The following is a solution first published in F. Viète's treatise "De emendatione", in 1615 [6, pp.184-5]. In the literature, it is genrally assumed that the leading coefficient is one, *i.e.*, $\phi_3 = 1$ (or one divides the polynomial by $\phi_3$ and relabel). A word of caution: if $\phi_3 \neq 1$ and is close to zero, we cannot immediately divide the polynomial by $\phi_3$ and must carry around this coefficient, to avoid numerical problems.

We are only interested in the real roots (three or one) here. Let us define the two quantities $Q$ and $R$ as:

$$\begin{aligned} Q &= \left(\frac{\phi_2}{3\phi_3}\right)^2 - \frac{\phi_1}{3\phi_3} = -\frac{\rho}{3} \\ R &= \left(\frac{\phi_2}{3\phi_3}\right)^3 - \frac{\phi_2\phi_1}{6\phi_3^2} + \frac{\phi_0}{2\phi_3} = \frac{\sigma}{2}. \end{aligned}$$

[4] Form the treatise *Arts Magna* ("The Great Art"), published by Girolamo Cardano, in 1545.

[5] For a quadratic, the discriminant is the well-know formula: $D = (b^2 - 4ac)/(2a)^2$.

[6] This uses the fact that for our reduced cubic, we have: $\overline{\Upsilon_1} + \overline{\Upsilon_2} + \overline{\Upsilon_3} = 0$, $\overline{\Upsilon_1\Upsilon_2} + \overline{\Upsilon_2\Upsilon_3} + \overline{\Upsilon_3\Upsilon_1} = \rho$, $\overline{\Upsilon_1\Upsilon_2\Upsilon_3} = -\sigma$.

[7] This was called the "casus irreducibilis" by Cardano *et al.*

In other words, we can rewrite the discriminant as:

$$D = 3^3 2^2 \left(Q^3 - R^2\right).$$

If $Q$ and $R$ are real, which is always the case for real coefficients, $\phi_1, \phi_2, \phi_3$, and $R^2 < Q^3$, then the cubic has *three real* roots obtained by the substitution:

$$\theta = \arccos\left(\frac{R}{\sqrt{-Q^3}}\right).$$

We derive the substitution in the cubic below. Then, the three real roots are given by:

$$\begin{aligned}
\rho_{y_1} &= 2\sqrt{-Q}\cos(\theta) - \left(\frac{\phi_2}{3\phi_3}\right) \\
\rho_{y_2} &= 2\sqrt{-Q}\cos\left(\frac{\theta + 2\pi}{3}\right) - \left(\frac{\phi_2}{3\phi_3}\right) \\
\rho_{y_3} &= 2\sqrt{-Q}\cos\left(\frac{\theta - 2\pi}{3}\right) - \left(\frac{\phi_2}{3\phi_3}\right).
\end{aligned}$$

Otherwise (*i.e.*, if $R^2 \geq Q^3$), we have a single real root. Compute:

$$A = -sgn(R)\,\left[|R| + \sqrt{R^2 - Q^3}\right]^{1/3}.$$

Next, compute:

$$B = \begin{cases} Q/A & (A \neq 0) \\ 0 & (A = 0) \end{cases}$$

in terms of which, the single real root is:[8]

$$\rho_{y_1} = (A + B) - \left(\frac{\phi_2}{3\phi_3}\right).$$

The explicit substitutions in the cubic $\Upsilon^3 + \rho\,\Upsilon + \sigma = 0$ to obtain the trigonometric solutions are as follows. First, use the substitution:

$$\Upsilon = \sqrt{\frac{4\,|\rho|}{3}}\,\overline{\Upsilon} = 2\sqrt{sgn(\rho)\,Q}\,\overline{\Upsilon},$$

which gives the cubic:

$$\overline{\Upsilon}^3 + \frac{3\rho}{4\,|\rho|}\overline{\Upsilon} = \left(\frac{3}{4\,|\rho|}\right)^{3/2}\sigma,$$

which becomes:

$$4\overline{\Upsilon}^3 + 3\,sgn(\rho)\,\overline{\Upsilon} = \frac{\sigma}{2}\left(\frac{3}{|\rho|}\right)^{3/2} = \frac{R}{\sqrt{-Q^3}} \equiv \cos\theta = \cos(3\varphi)\,,$$

where $\varphi = \theta/3$. Next, observe that by de Moivre's formula, we have:

$$4\cos^3\varphi - 3\cos\varphi = \cos(3\varphi).$$

By identitifying the last two equations, we get the three roots in $\overline{\Upsilon}$ :

[8] The two complex (conjugate) roots are derived from the same factors; see [6, pp.185] for details.

$$\overline{\Upsilon_i} = \cos(\varphi + 2n\pi) = \cos\left(\frac{\theta + 2n\pi}{3}\right) = \cos\left[\frac{1}{3}\arccos\left(\frac{R}{\sqrt{-Q^3}}\right) + \frac{2n\pi}{3}\right] ,$$

where $n = -1, 0, 1$ and we have the constraints $\rho < 0$ and $|3\varphi| = |\theta| \leq 1$ and thus $R^2 < Q^3$. Then, we have the three roots in $\Upsilon$:

$$\Upsilon_i = 2\sqrt{-Q}\,\cos\left(\frac{\theta + 2n\pi}{3}\right) .$$

And, finally, the back-substitution, $\rho_y = \Upsilon - (\phi_2/3\phi_3)$, give us the roots of the original cubic.

## 2.2 Newton's Iterative Solution

In Newton's method, we construct an "iterator" from the tangent to the function of interest. In the case of our cubic:

$$f(\rho_y) = \phi_3\,{\rho_y}^3 + \phi_2\,{\rho_y}^2 + \phi_1\,\rho_y + \phi_0 ,$$

we have the following iterator [5, p.28]:

$$F(\rho_y) = \frac{2\phi_3{\rho_y}^3 + \phi_2{\rho_y}^2 - \phi_0}{3\phi_3{\rho_y}^2 + 2\phi_2\rho_y + \phi_1} ,$$

with the iteration rule:

$${\rho_y}_{i+1} = F({\rho_y}_i) .$$

The fixed point of such an iterator (where its graph is flat) correspond to zeros of the associated polynomial equation. The iterator itself, is derived from Newton's formula:

$$F(\rho_y) = \rho_y - \frac{f(\rho_y)}{df(\rho_y)} ,$$

where *df* denotes the derivative of $f$ with respect to the explicit variable, $\rho_y$. Note that the above iterative solution requires an initial value.

# Chapter 3

# Solving Quartics

Consider the ***quartic*** implicit polynomial in terms of $\rho_y$ ,

$$p^{(4)}(\rho_y) = \phi_4\,{\rho_y}^4 + \phi_3\,{\rho_y}^3 + \phi_2\,{\rho_y}^2 + \phi_1\,\rho_y + \phi_0 = 0,$$

where $\phi_4 \neq 0$, or in simplified form:

$${\rho_y}^4 + a{\rho_y}^3 + b{\rho_y}^2 + c\rho_y + d = 0.$$

We first divide this equation by $\phi_4$, and replace the variable $\rho_y$ by:

$$\rho_y = \Upsilon - (\phi_3/4\phi_4)\ , \tag{3.1}$$

in order to get the simplified expression (without the cubic term in $\rho_y$):

$$\Upsilon^4 + p\Upsilon^2 + q\Upsilon + r = 0\ , \tag{3.2}$$

where the coefficients $p$, $q$ and $r$ are defined as:

$$\begin{aligned}
p &= \frac{\phi_2}{\phi_4} - \frac{3\,\phi_3^2}{8\,\phi_4^2} = b - \frac{3}{8}a^2\,, \\
q &= \frac{\phi_1}{\phi_4} - \frac{\phi_2\,\phi_3}{2\,\phi_4^2} + \frac{\phi_3^3}{8\,\phi_4^3} = c - \frac{1}{2}ab + \frac{1}{8}a^3\,, \\
r &= \frac{\phi_0}{\phi_4} - \frac{\phi_1\,\phi_3}{4\,\phi_4^2} + \frac{\phi_2\,\phi_3^2}{16\,\phi_4^3} - \frac{3\,\phi_3^4}{256\,\phi_4^4} = d - \frac{1}{4}ac + \frac{1}{16}a^2b - \frac{3}{256}a^4\,.
\end{aligned}$$

Let us first consider two special cases. If $r = 0$, then the roots are given by $\rho_y = -\phi_3/4\phi_4$ and by the roots of the factored cubic $\Upsilon^3 + p\Upsilon + q = 0$, plus $\phi_3/4\phi_4$ . If $q = 0$, then the quartic is in fact a double-quadratic in $\Upsilon^2$, for which we take the signed root of each of the two computed roots, to get four roots in $\Upsilon$, which, when added with $\phi_3/4\phi_4$, give the four roots in $\rho_y$. In general however, further substitutions are needed, leading to many different solutions in the literature. They all boil down to first find an auxiliary cubic, also called “resolvent” or “subsidiary” cubic. We will study some of these in the following sections.

## 3.1 Auxiliary Cubics

The auxiliary cubics we found in the literature are the following

- Ferrari-Lagrange [4]:

$$w^3 + b\,w^2 + (ac - 4d)\,w + (a^2d + c^2 - 4bd) = 0\ .$$

- Cardano-Descartes-Euler [4]:

$$w^3 + 2p\, w^2 + (p^2 - 4r)\, w - q^2 = 0 \ ,$$

where, expanding the coefficients in terms of $a, b, c$, we have:

$$\begin{aligned} p^2 - r &= \frac{3}{16}a^4 - a^2 b + ac + b^2 - 4d \\ q^2 &= \frac{1}{64}a^6 - \frac{1}{8}a^4 b + \frac{1}{4}a^3 c + \frac{1}{4}a^2 b^2 - abc + c^2 \end{aligned}$$

- Neumark [4]:

$$w^3 - 2b\, w^2 + (ac + b^2 - 4d)\, w + (a^2 d - abc + c^2) = 0 \ .$$

Thus, we have various cubics, the coefficients of which are derived from those of the original quartic. In most methods, *one* root of the cubic is then used to factorize the quartic into a pair of quadratics. Stable combination of signs of the coefficients, $a, b, c, d$, the cubic root and the coefficients of the intermediary quadratics, have been studied in [4]. There it is shown that no single choice of an auxiliary cubic is stable in all cases. On average, however, it appears that Neumark's and Ferrari's methods are to be preferred over Cardano's.[1] Still, in some less frequent cases (combinations of signs) Cardano's has a better behavior.

In the following, we mention an alternative to Ferrari's method of completing the squares, given one initial root of any of the auxiliary cubics listed above, which was discovered by Euler and which uses all three roots of the auxiliary cubic (Cardano's). Then, we summarize Ferrari's method. See [4] for the detailed discussion on the three auxiliary cubics' stability behavior.

### 3.1.1 The Euler-Galois Resolvent Cubic

Define the auxiliary cubic to equation (3.2) as (with the substitution, $\rho_y = 4w$):

$$w^3 + \frac{p}{2}w^2 + \left(\frac{p^2 - 4r}{16}\right) w - \frac{q^2}{64} = 0 \ .$$

Next, find the roots, $w_1, w_2, w_3$, of this cubic (§ 2). Note that, none of these is zero, because the factor $q \neq 0$. Let $\varpi_1^2 = w_1$ and $\varpi_2^2 = w_2$ be the squares of two of the three roots. Then, define the following value:

$$\varpi_3 = -\frac{q}{8\varpi_1 \varpi_2} \ .$$

Euler showed that $\varpi_3^2 = w_3$, the third root of the auxiliary cubic. He also showed that the four roots of the original quartic can then be obtained as [2, p.121]:

$$\begin{aligned} \rho_{y_1} &= \varpi_1 + \varpi_2 + \varpi_3 - \frac{\phi_3}{4} \\ \rho_{y_2} &= \varpi_1 - \varpi_2 - \varpi_3 - \frac{\phi_3}{4} \\ \rho_{y_3} &= -\varpi_1 + \varpi_2 - \varpi_3 - \frac{\phi_3}{4} \\ \rho_{y_4} &= -\varpi_1 - \varpi_2 + \varpi_3 - \frac{\phi_3}{4} \ . \end{aligned}$$

In the following we derive explicitly how the resolvent cubic and the above formula for the roots of the quartic can be obtained.

[1] Note that there are more terms, and of higher degrees, for the coefficients of the auxiliary cubic of Cardano than for Ferrari's or Neumark's.

**Galois solution of the general quartic polynomial**

Let us denote the roots of the reduced quartic, $\Upsilon^4 + p\Upsilon^2 + q\Upsilon + r = 0$, by $y_1, y_2, y_3, y_4$. We can thus re-write the reduced quartic as:

$$p^4(\Upsilon) = (\Upsilon - y_1)(\Upsilon - y_2)(\Upsilon - y_3)(\Upsilon - y_4) = 0 \ .$$

Note that, because this quartic has no third degree term, the sum of the roots must be zero:

$$y_1 + y_2 + y_3 + y_4 = 0 \ .$$

Now, define the following values:

$$\begin{aligned} \varpi_1 &= (y_1 + y_2)(y_3 + y_4) \\ \varpi_2 &= (y_1 + y_3)(y_2 + y_4) \\ \varpi_3 &= (y_1 + y_4)(y_2 + y_3) \ . \end{aligned}$$

Then, the polynomial:

$$\begin{aligned} p^3(y) &= (y - \varpi_1)(y - \varpi_2)(y - \varpi_3) \\ &= y^3 - (\varpi_1 + \varpi_2 + \varpi_3)\, y^2 + (\varpi_1\varpi_2 + \varpi_1\varpi_3 + \varpi_2\varpi_3)\, y - \varpi_1\varpi_2\varpi_3 \ , \end{aligned}$$

is the cubic resolvent of $p^4(\Upsilon)$, and a little calculations show that [3]:

$$p^3(y) = y^3 - 2py^2 + (p^2 - 4r)y + q^2 \ .$$

### 3.1.2 Ferrari's method: Completing the Squares

We can factorize eqn. (3.2) into two perfect squares $\Gamma^2 - \Theta^2 = (\Gamma + \Theta)(\Gamma - \Theta) = 0$, where $\Gamma = \Upsilon^2 + u/2$, for any $u \in I\!R$. This is achieved by adding and subtracting $\Upsilon^2 u + u^2/4$ to eqn. (3.2). For $\Theta$ we thus have:

$$\Theta^2 = (u - p)\Upsilon^2 - q\Upsilon + \left(\frac{1}{4}u^2 - r\right) \ .$$

$\Theta^2$ is a perfect square for those $u$ such that:

$$q^2 = 4(u - p)\left(\frac{1}{4}u^2 - r\right) \ .$$

This is a *resolvent cubic*, and plugging a solution, $u_1$, back in gives us:

$$\Gamma^2 - \Theta^2 = (\Gamma + \Theta)(\Gamma - \Theta) \ .$$

Thus, we can now rewrite eqn. (3.2) to be factorized as:

$$(\Upsilon^2 + u_1/2 + \Theta)\,(\Upsilon^2 + u_1/2 - \Theta) = 0 \qquad (3.3)$$

where:

$$\begin{aligned} \Theta &= \Lambda\,\Upsilon - \Delta \\ \Lambda &= \sqrt{u_1 - p} \\ \Delta &= -q/(2\Lambda) \end{aligned}$$

Furthermore, since the coefficients are all real numbers, one can show that at least one real number $u_1 \geq p$ satisfies the above equation.[2] The roots of eqn. (3.2) are the same as those of the above two quadratic factors in eqn. (3.3). Let us rewrite the two quadratic factors explicitly in terms of powers of $\Upsilon$:

$$\begin{aligned} \Gamma + \Theta &= \Upsilon^2 + \Lambda\,\Upsilon + \left(\frac{u_1}{2} - \Delta\right) = 0 \\ \Gamma - \Theta &= \Upsilon^2 - \Lambda\,\Upsilon + \left(\frac{u_1}{2} + \Delta\right) = 0 \end{aligned}$$

Then, we simply complete the squares to find the roots of each quadratic factors above, e.g. [1, p.119]:

$$\begin{aligned} \Upsilon_{1,2} &= \frac{1}{2}\left(-\Lambda \pm \sqrt{\Lambda^2 - 2u_1 + 4\,\Delta}\right) \\ \Upsilon_{3,4} &= \frac{1}{2}\left(\Lambda \pm \sqrt{\Lambda^2 - 2u_1 - 4\,\Delta}\right) \end{aligned}$$

Note that, a better (numerically speaking) solution for quadratic factors is as presented above.

[2] See [1, §5.6] for more details.

# Bibliography


[1] Garrett Birkhoff and Saunders Mac Lane. *A Survey of Modern Algebra.* Macmillan, New York, fourth edition, 1977. 500 pages.

[2] I. N. Bronshtein and K. A. Semendyayev. *Handbook of Mathematics.* Van Nostrand Reinhold, New York, NY, U.S.A., $3^{rd}$ edition, 1985. English version.

[3] W. M. Faucette. A geometric interpretation of the solution of the general quartic polynomial. *American Mathematical Monthly*, 103:51–57, January 1996.

[4] Don Herbison-Evans. Solving quartics and cubics for graphics. Technical Report TR94-487, Basser Dept. of Computer Science, University of Sydney, Australia, 2000. Updated, July 26, 2000.

[5] Kurth Kreith and Don Chakerian. *Iterative Algebra and Dynamic Modeling.* Textbooks in Mathematical Sciences. Springer, NY, 1999. 325 pages.

[6] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. *Numerical Recipes in C.* Cambridge University Press, Cambridge, U.K., $2^{nd}$ edition, 1992. 994 pages.