The Shock Scaffold

— A directed graph hierarchy for the 3D medial axis and shape[1]

The Symmetry Set ($\mathcal{SS}$) is the closure of the locus of centers of spheres tangent to smooth surface patches in two or more loci; such bitangent spheres are called "contact spheres." The Medial Axis ($\mathcal{MA}$) is the subset of the $\mathcal{SS}$ for which all such spheres are *maximal*, *i.e.*, such that no other contact spheres are contained in them.[2]

A classification of $\mathcal{MA}$ points was introduced by Giblin and Kimia [6, 7], which we now summarize. Let A_k^n denote a circle (in 2D) or a sphere (in 3D) osculating a boundary element at n distinct points, each with $k+1$ degree of contact, Figure 1: $k = 1$ denotes regular tangency; $k = 2$ denotes a sphere of curvature for a surface patch; $k = 3$ denotes a sphere of curvature at a *ridge* point; $k = 4$ denotes a sphere of curvature at a turning point of a ridge, *etc.* [8, Ch.6]. Only odd orders of contact (*i.e.*, $k = 1, 3$) can contribute to a $\mathcal{MA}$ type of shock, that is, as being the center of a *maximal* sphere. Then, a classification based on the number and order of contact [6] leads to ***five*** principal types of shock points: A_1^2, A_1^3, A_3, A_1^4 and A_1A_3 (Figure 1).[3]

1. A_1^2 *contact*: this is a sphere with two ordinary A_1 contacts. The local form of the A_1^2 is such that the centers of the contact spheres trace a surface, called *sheet*, which is locally smooth.

2. A_3 *contact*: this is the limiting case of two A_1^2 points which come together; it corresponds in 2D to the center of curvature at a curvature extrema and in 3D to *rib* curves associated to *ridges* on the boundary.

3. A_1^3 *contact*: the contact sphere has three ordinary A_1 contacts. The local form is one where three sheets come together at a *curve, i.e.*, choosing any 2 of these 3 tangency points and moving the sphere so that it remains bitangent to the bounding surface at points close to these two, results in a smooth sheet of the $\mathcal{SS}$ or $\mathcal{MA}$ for each pair.

4. A_1A_3 *contact*: it contains the centers of spheres which have contact with the surface in two places, one near the original A_1 point (*i.e.*, ordinary tangency) and one near the A_3 rib point. Furthermore, at an A_1A_3 point, an A_1^3 curve also "terminates" together with the A_3 curve.

5. A_1^4 *contact*: the contact sphere has four ordinary contacts, which is generic, *i.e.*, four points in space determine a unique sphere (such that they are not co-linear nor co-circular). At the center of the sphere passes *six* smooth *sheets* of the $\mathcal{MA}$ (*i.e.*, 6 distinct pairs from 4 contact points). An alternative view of this event, is as the combination/intersection of *four* A_1^3 *curves* (*i.e.*, 4 distinct triplets from 4 contact points).

Two observations are significant here. First, the topology of each of these types is as follows: A_1^2 points are interior points of a medial surface, called "sheet;" A_3 points organize into curves representing *ridges* on surfaces and are the "exterior" boundary of $\mathcal{MA}$ sheets, called "ribs" or "skeletal edges;" A_1^3 points organize into curves which are the intersection of three A_1^2 sheets — these curves often correspond to "generalized axis" as well as to "interior" boundary of $\mathcal{MA}$ sheets, and are sometimes called "seams" or "axial curves;" A_1^4 and A_1A_3 are isolated points where four A_1^3 or a pair of A_1^3 and A_3 curves intersect, respectively

[1] Extract from Frederic Fol-Leymarie's Ph.D. thesis [9, Ch.3].

[2] Contact with isolated input points is taken as the limit of a contact with tiny spheres having radii shrinking to zero. The maximality criterion is equivalent to "emptyness," *i.e.*, a maximal contact sphere is such that it contains no other input points.

[3] This notation corresponds to the one used to describe singularity varieties of minima functions of three variables in the Singularity Theory of Dynamical Systems, *e.g.*, see the works of Arnold [1]. The "A" comes from the relation to the simple Lie groups of type A.

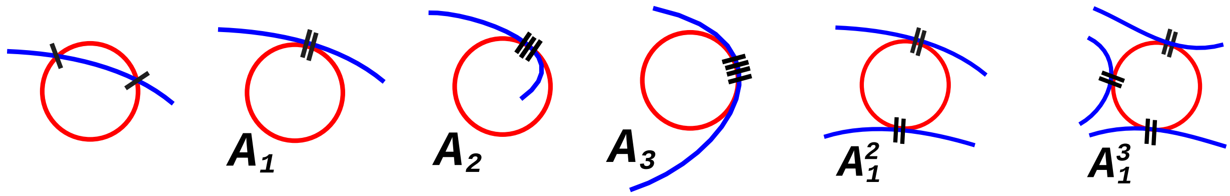


Figure 1: Illustration of the notation A_k^n based on contact of a curve with a circle (from [6]). $k+1$ counts order or degree of contact (indicated by straight short dark segments): A_1 is regular tangent contact, A_2 is regular "curvature" contact, A_3 is a curvature maximum contact. The superscript n counts the number of contact points, so that A_1^2 means two A_1 contacts. A similar definition holds for the contact of surfaces with spheres.

Shock flow	1	2	3	4
Sheet	$A_1^2\!-\!1$	$A_1^2\!-\!2$	$A_1^2\!-\!3$	$A_1^2\!-\!4$
Ridge	$A_3\!-\!1$	$A_3\!-\!2$	$A_3\!-\!3$	$A_3\!-\!4$
Axis	$A_1^3\!-\!1$	$A_1^3\!-\!2$	$A_1^3\!-\!3$	$A_1^3\!-\!4$
Ridge end	-	$A_1A_3\!-\!2$	$A_1A_3\!-\!3$	$A_1A_3\!-\!4$
Axis end	-	$A_1^4\!-\!2$	$A_1^4\!-\!3$	$A_1^4\!-\!4$

Table 1: Final classification of 18 possible shock points based on contact with spheres, A_k^n, and flow type.

The *shock structure* arises from a "dynamic" interpretation of the $\mathcal{MA}$, as the locus of singularities — or shocks — formed in the course of wave propagation from boundaries with associated direction and speed of flow — as in Blum's grassfire [4]. The flow for each $\mathcal{MA}$ point is defined in the direction of increasing radius, r, of associated maximal contact spheres in a neighborhood of that point. *Flow* is thus a *vector field*, taken as the projection of the gradient of r on the $\mathcal{MA}$: $\nabla r \cdot T$, where T defines the tangent space to the $\mathcal{MA}$ sheets and curves. This flow itself can have singularities, and shocks thereby can "flow" along sheets (A_1^2) or curves (A_3 and A_1^3) in various ways, as summarized below.

Regular shock (or 1st order): A shock point at which flow goes through smoothly: *(i)* along a sheet: $A_1^2\!-\!1$; *(ii)* along a curve: $A_1^3\!-\!1$, $A_3\!-\!1$.

Shock source (2nd order): A shock which initiate flow: *(i)* along a sheet: $A_1^2\!-\!2$; *(ii)* along a curve: $A_1^3\!-\!2$, $A_3\!-\!2$; *(iii)* at a vertex: $A_1A_3\!-\!2$.

Shock relay (3rd order): A shock which is both a source and sink for the flow: *(i)* for a sheet: $A_1^2\!-\!3$; *(ii)* for a curve: $A_1^3\!-\!3$, $A_3\!-\!3$; *(iii)* for a vertex: $A_1^4\!-\!2$, $A_1^4\!-\!3$, $A_1A_3\!-\!3$.

Shock sink (4th order): A shock at which flow type terminates: *(i)* for a sheet: $A_1^2\!-\!4$; *(ii)* for a curve: $A_1^3\!-\!4$, $A_3\!-\!4$; *(iii)* for a vertex: $A_1^4\!-\!4$, $A_1A_3\!-\!4$.

This classification of the $\mathcal{MA}$ into *eighteen* types of shock points (Table 1) leads to a powerful graph structure for its representation, where *regular shock points need not be traced explicitly*.

Our goal is to propose a small set of representations built from the MA and to make explicit a 3D graph whose nodes are taken from the set of 15 types of shock singularities, *i.e.*, sources, relays and sinks, and whose links connect the selected nodes. We start with a general *hypergraph* which includes all special points as nodes, special curves as links, and sheets as hyperlinks. We then present coarser versions where hyperlinks have been removed, therefore leaving a graph which is the shock scaffold proper, *i.e.*, made of shock point singularities as nodes, and linked by curve segments forming in space a structure resembling the scaffoldings used to erect buildings. We follow standard definitions of hypergraphs and graphs from the literature (*e.g.*, see [2, 3]); which are constructed from the pair "nodes and (hyper)links." First, we define the elements we will use to construct the various representations in the hierarchy, *i.e.*, nodes, links, and hyperlinks.

Definition 1 (Shock nodes, P) *The set of* shock nodes*, denoted P, is comprised of shock sources, relays*[4] *and sinks for shock curves and vertices,* i.e.*: $A_1^3{-}2$, $A_1^3{-}3$, $A_1^3{-}4$, $A_3{-}2$, $A_3{-}3$ and $A_3{-}4$ points for curves, and all remaining types of A_1A_3 and A_1^4 points for vertices.*

Definition 2 (Shock (curve) links, L) *A* shock link*, L, for each curve segment between two shock nodes is an* ordered *(by the radius function) pair of these two shock nodes, and it has attributes describing its geometry and dynamics.*

Definition 3 (Shock (sheet) hyperlinks, H) *A* shock hyperlink *H for each sheet is the* ordered, cyclic *set of shock nodes of its associated bounding curves and vertices. A hyperlink is attributed with geometry and dynamics of the sheet.*

Note that a hyperlink therefore gives an orientation to the shock sheet. We can now define the first level in our hierarchy, which augments the $\mathcal{MA}$ with a directed graph structure.

Definition 4 (Augmented shock scaffold, $\mathcal{SC}^+$) *The* augmented shock scaffold*, denoted $\mathcal{SC}^+$, is the $\mathcal{MA}$ augmented with the set of shock nodes, P, connected by links L and hyperlinks H.*

The advantage of the augmented graph structure over the ("classical") trace of the $\mathcal{MA}$ is that it organizes the $\mathcal{MA}$ information into groups and specifies their connectivity. It is precisely the connectivity among these groups which contains the qualitative information, while the remaining information allows for an exact reconstruction or an approximation of the shape from the shock hypergraph [5]. If we drop from this $\mathcal{SC}^+$ the hyperlinks, H, which contain the explicit representation of the sheets and their interior, we are left with an "ordinary" graph structure which defines the connectivity among the retained shock nodes via explicit links only. This graph summarizes the $\mathcal{MA}$ (Figure 2).

Definition 5 (Shock scaffold, $\mathcal{SC}$) *The* shock scaffold *is a geometric directed graph, denoted $\mathcal{SC}$, with nodes P and links L.*

In general, the shock scaffold will be *connected*, *i.e.*, each node in P will be reachable from other nodes via a *chain*, *i.e.*, a succession of links where we allow navigation against the radius flow.[5] When the initial data consists of closed boundaries delimiting compact objects, the shock scaffold will be broken into separate sub-graphs for the interior of each such object as well as for the exterior region; each sub-graph will be connected. The shock scaffold is not a tree in general, *i.e.*, it contains *circuits* (chains of links forming closed loops).

[4] Note that relays for the flow which correspond to degeneracies — *i.e.*, at which the flow travels at infinite speed — can be represented by an arbitrarily selected point along this shock "relay."

[5] The shock scaffold is not *strongly connected* [10, p.29] in general, *i.e.*, there exists no *directed* paths between certain pairs of nodes.

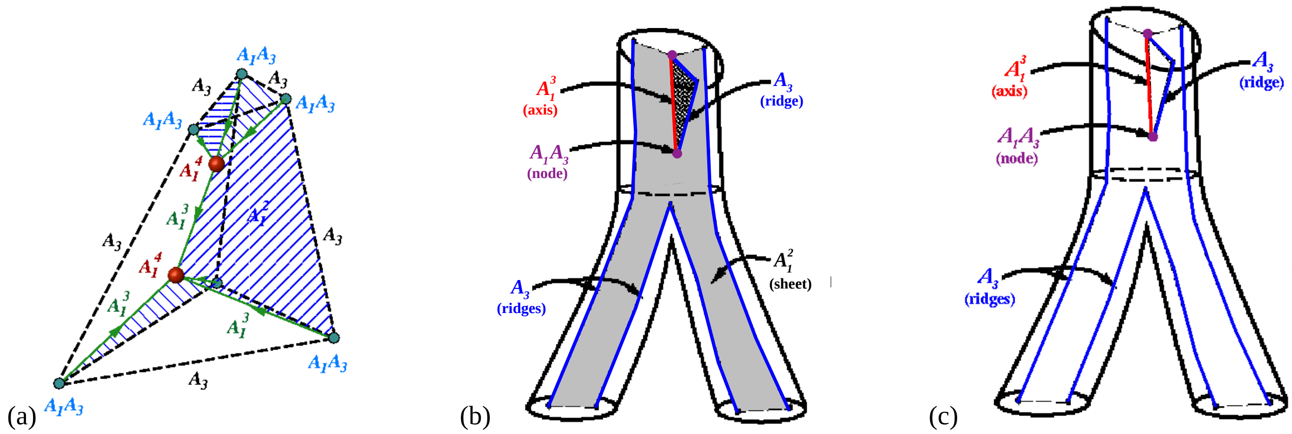


Figure 2: The 3D *augmented shock scaffold*, $\mathcal{SC}^+$, is illustrated for a truncated tetrahedron, which has 8 nodes, 7 links and 9 hyperlinks. The dark broken lines are surface ridges (A_3), the smaller dots are surface vertices (A_1A_3), the larger nodes are A_1^4 shocks, the interior links have arrows to indicate flow (all A_1^3's here), the hashed sheets are hyperlinks (A_1^2; not all shown). Sketches of $\mathcal{SC}^+$ in (b) and $\mathcal{SC}$ in (c) for a branching structure which at the top is a cylinder whose base grows from a triangle to an ellipse, and which splits into two cylindrical structures with elliptic bases (only the hyperlink interior to the shape is shown).

Despite the lack of an explicit representation of sheets, from the shock scaffold alone we are still able to get a fairly good idea of the shape of the object due to the remaining connectivity (Figure 5). The $\mathcal{MA}$ can be approximated by interpolating the missing $\mathcal{MA}$ sheet points, by stretching smooth elastic surfaces over the links L, much as is done when a "tent" is constructed from its scaffold. If we also make the representation of shock curves implicit, we obtain a simpler graph.

Definition 6 (Reduced shock scaffold) *The* reduced shock scaffold, *denoted* $\mathcal{SC}^-$, *is the* $\mathcal{SC}$ *where link attributes (*i.e., *geometry and dynamics) have been discarded.*

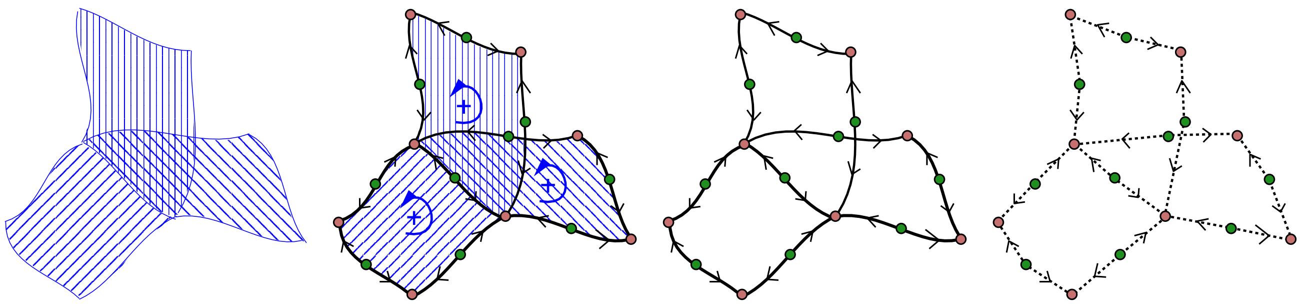
The reduced and "ordinary" shock scaffolds have common (graph) properties (of being connected, *etc.*). This three-tier hierarchical representation for the $\mathcal{MA}$, where $\mathcal{SC}^+ \supset \mathcal{SC} \supset \mathcal{SC}^-$, is illustrated in Figure 3.[6]

We also note that *at the very coarsest level* ***only connectivity amongst nodes*** *need be retained*. That is, we could do away with the geometry of nodes and define a strictly abstract graph, where a node is simply a representative of a sheet, curve or vertex. We call this representation, void of geometry, the *topological scaffold*, and denote it $\mathcal{TS}$ (Figure 4.(b)).[7]

Finally, we can further characterize shock sheets' interior by building a network connecting their nodes together with the nodes at the boundaries of sheets, *i.e.*, with nodes of bounding curves and vertices, thereby defining a "full" shock hypergraph, denoted $\mathcal{SH}$ (Figure 4.(a)).

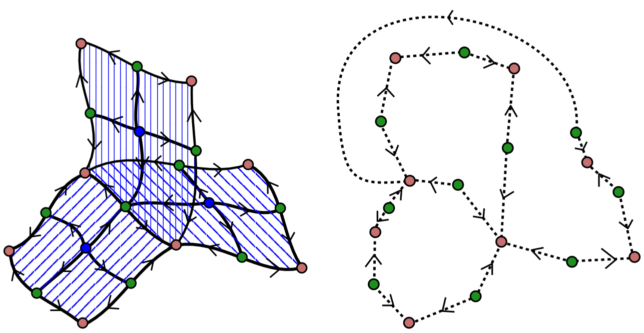
[6]Note that, although we represent the classical view of the $\mathcal{MA}$ without distinguishing medial curves and vertices, some authors do define the $\mathcal{MA}$ with an explicit identification of these, particularly in the domain of CAD (*e.g.*, see [12, 11]).

[7]Note however, that we use the geometry of shock nodes in the construction of the shock scaffold itself [9, Ch.4]. And then, the geometry of nodes proves also useful in applications such as surface recovery [9, Ch.6].



(a) $\mathcal{MA}$ (b) $\mathcal{SC}^+$ (c) $\mathcal{SC}$ (d) $\mathcal{SC}^-$

Figure 3: From the "classical" $\mathcal{MA}$ static representation to the *shock scaffold*. (a) Typical situation in 3D, where three medial sheets intersect into a medial curve. (b) Equivalent representation by the *augmented shock scaffold*, where shock nodes along curves are connected by directed links; hyperlinks cyclic order is indicated by a counterclockwise arrow. (c) Representation by the *shock scaffold*, where the interior of sheets is implicit. (d) Representation by the *reduced shock scaffold*, where the trace of shock sheets and curves is implicit. Red points correspond to shock (or $\mathcal{MA}$) vertices, *i.e.*, A_1^4 or A_1A_3. Green points correspond to shock sources of curves, *e.g.*, $A_1^3{-}2$ points.



(a) $\mathcal{SH}$ (b) $\mathcal{TS}$

Figure 4: Possible additional levels in the shock scaffold hierarchy. (a) The *shock hypergraph* adds to $\mathcal{SC}$ sources, relays and sinks of shock sheets (indicated as blue dots) and links amongst these as well as with respect to the sheet boundaries. (b) The *topological scaffold* is obtained from the $\mathcal{SC}$ when only the topology of the graph structure is preserved.

Level	Symbol	Features
I	$\mathcal{SH}$	All shock flow singularities, connected via links and hyperlinks.
II	$\mathcal{SC}^+$	Set of nodes P, links L, and hyperlinks H.
III	$\mathcal{SC}$	Directed graph $\{P, L\}$
IV	$\mathcal{SC}^-$	Set of nodes P, and links stripped of geometry.
V	$\mathcal{TS}$	Set of nodes and links stripped of geometry.

Table 2: The complete scaffold hierarchy is comprised of five levels.

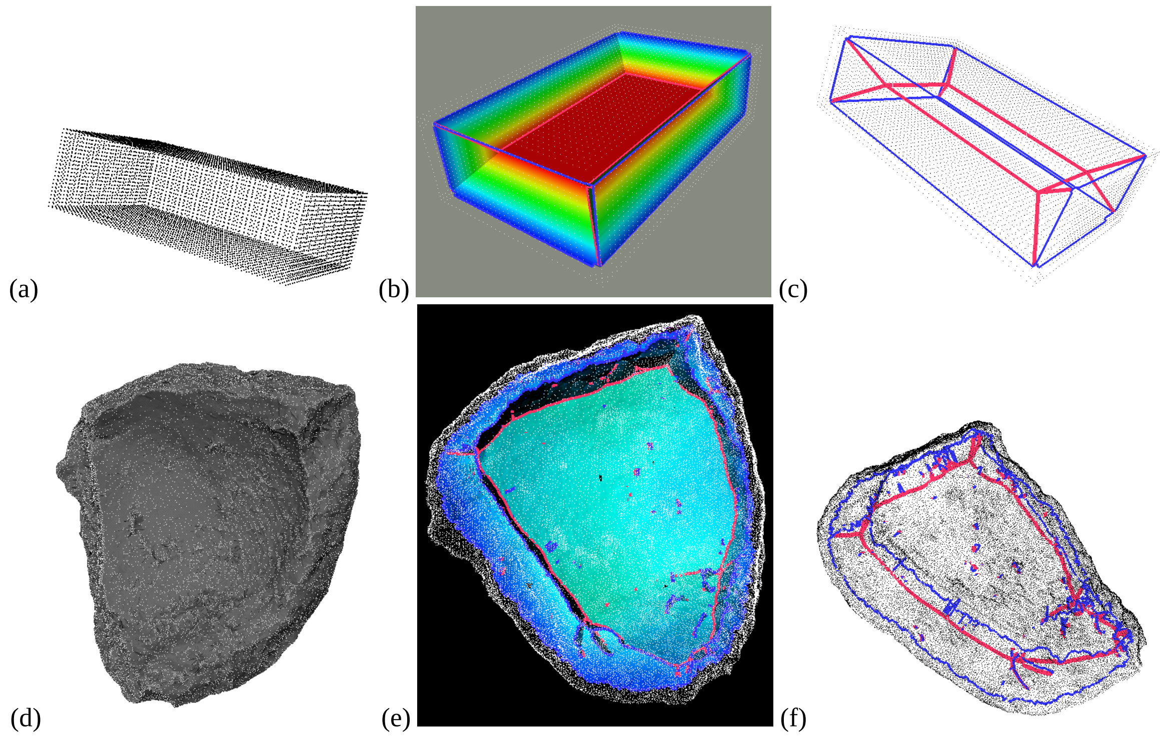


Figure 5: The augmented shock scaffold, $\mathcal{SC}^+$, of a rectangular box sampled by 7326 points (a) is depicted in (b). The flow along sheets is shown using the color spectrum, where blue means close to the boundary, and red means as far as possible. In (c) the geometry for the interior of the shock sheets is left implicit, and (A_1^3) axial curves at the intercepts of shock sheets are shown in pink, while (A_3) ribs at the boundaries of shock sheets are shown in blue. This synthetic example serves as a prototype of many real shapes, such as the pot sherd in (d) which can be thought of as a deformed rectangular box with additional surface perturbations (approximatively 40.000 point samples here, obtained by laser scanning). $\mathcal{SC}^+$ of this sherd is shown in (e) with the flow along sheets color-coded similarly to (b) where the missing colors of the spectrum correspond to the symmetries away from the concave part of the pot sherd (not shown here); white dots indicate input data. In (f) is shown the corresponding $\mathcal{SC}$ (input point samples in black).

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