Creative Computing II

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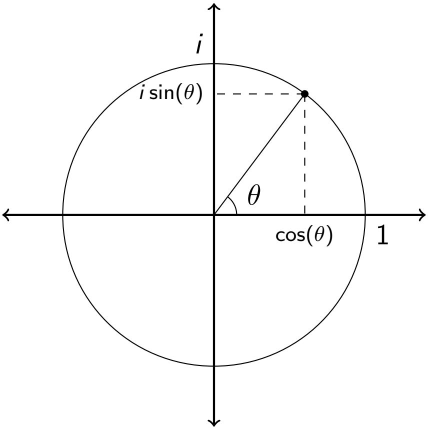
Autumn 2009, Tuesdays, 10:00–15:00
Winter 2010, Tuesdays, 13:30–17:30

Signals

The Complex Exponential



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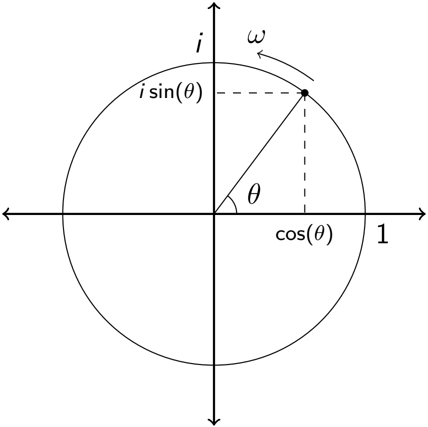


$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t);\ e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t).$$

Signals

Sinusoidal functions

Functional relations:

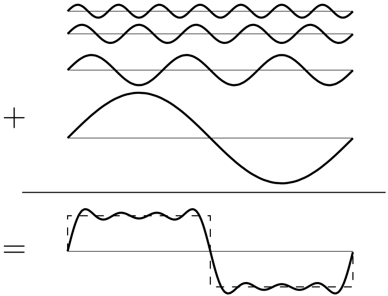
- $\cos(\omega t) = \frac{e^{i\omega t}+e^{-i\omega t}}{2}$
- $\sin(\omega t) = \frac{e^{i\omega t}-e^{-i\omega t}}{2i}$

Identities:

- $e^{i\pi} = -1$ (Euler's Identity)
- $e^{i\frac{\pi}{2}} = i$
- $e^{2\pi i} = 1$
- $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$ (de Moivre's formula)

Square wave:

$$s_f(t) = \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin(2\pi(2k-1)ft)$$



Fourier Series:

- Any signal can be written as a weighted sum of sin and cos terms: a **Fourier Series**.
- For a signal of length L, all sinusoids have angular frequencies that are integer multiples of $\frac{2\pi}{L}$.
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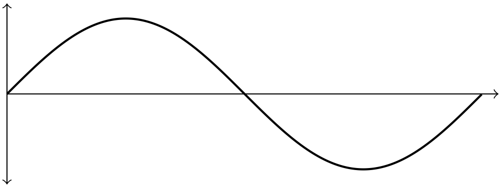
Fourier Analysis of Signals:

- Extraction of frequency components for a given signal;
- Dot-product multiply by complex exponential signal;
- Magnitude and phase of result give magnitude and phase of corresponding sinusoid.

How does this work?

- dot-product of sinusoid with *exactly itself* gives a non-zero result;
- all other dot-products between sinusoids give zero.
- sinusoids are **orthogonal** basis functions.



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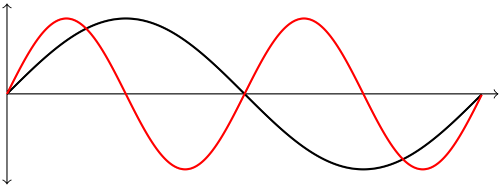

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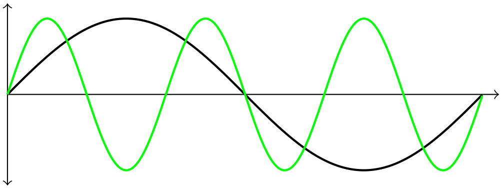

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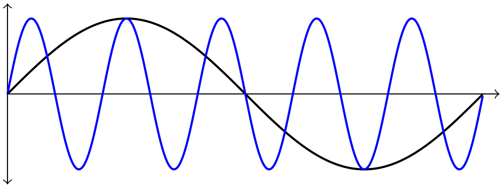

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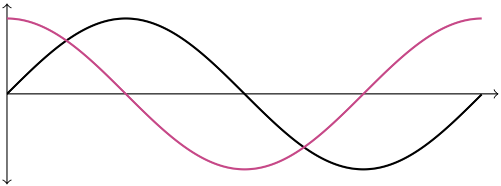

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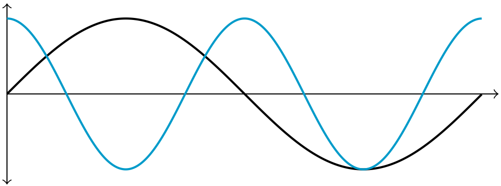

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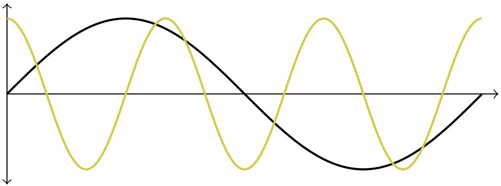

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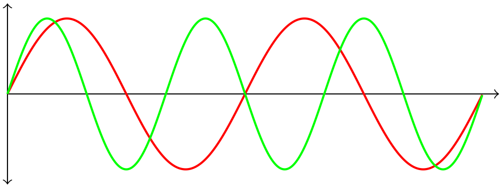

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Fourier Transforms

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Note: $\mathcal{F}(x)$ sometimes notated as $\tilde{x}$.

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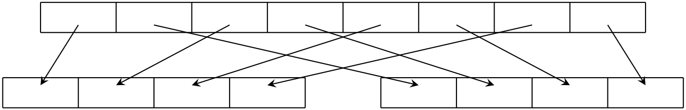

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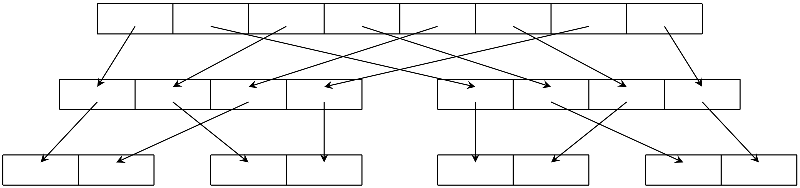

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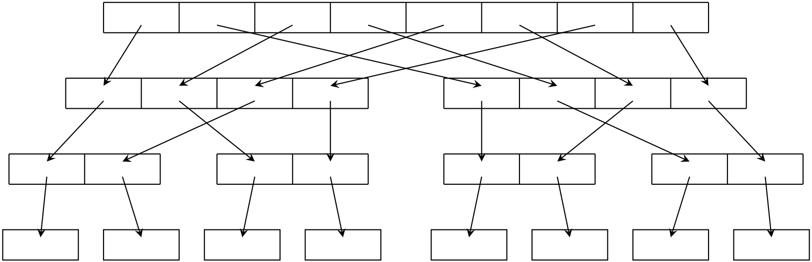

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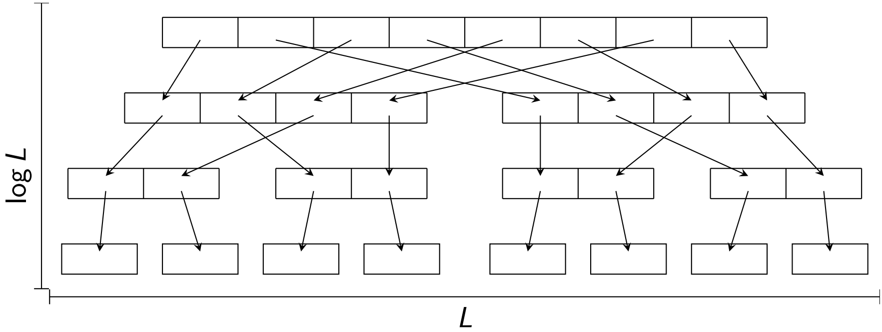

Overall complexity $O(L \log L)$

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Fourier Transforms and Convolution

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Octave:

- `ifft(fft(h,length([h x])-1).*fft(x,length([h x])-1))`

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$$y = \mathcal{F}^{-1}(\mathcal{F}(h) \times \mathcal{F}(x))$$

so

$$\mathcal{F}(y) = \mathcal{F}(h) \times \mathcal{F}(x)$$

- $\mathcal{F}(h)$ is the **frequency response** of the system.
- the frequency spectrum of the output signal is the product of the spectrum of the input and the frequency response of the system.